Coherence and finiteness spaces

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Abstract. This very short note presents an unsuspected relation between coherence spaces from [4] and finiteness spaces from [2] in the form of a linear functor from **Coh** to **Fin**. The main used in this analysis is the infinite Ramsey theorem.

0. Introduction. The category of coherence spaces was the first denotational model for linear logic (see [4]): the basic objects are reflexive, non oriented graphs; and we are more specifically interested by their *cliques* (complete subgraph). If C is such a graph, we write $\mathcal{C}(C)$ for the collection of its cliques.

Coherence spaces enjoy a very rich algebraic structure where the most important operations on them are:

- taking the (reflexive closure of the) complement (written $C_1^\perp$);
- taking a cartesian product (written $C_1 \otimes C_2$);
- taking a disjoint union (written $C_1 \oplus C_2$).

If one only looks at the vertices, the corresponding operations are simply the identity, the usual cartesian product "$\times$" and the disjoint union "$\oplus$".

More recently, Thomas Ehrhard introduced the notion of finiteness spaces ([2]) to give a model to the differential λ-calculus ([3]), which can be seen as an enrichment of linear logic. The point that interests us most here is that the collection of finitary sets of a finiteness space are closed under finite sums (*i.e.* finite unions) to take into account a notion of "non-deterministic sum" of terms. (See also [5].) This is definitely not possible with the cliques of a coherence space.

Very briefly, a finiteness space is given by a set $|\mathcal{F}|$, called the *web*, and a collection $\mathcal{F}$ of subsets of $|\mathcal{F}|$ such that

$$\mathcal{F}^{\perp\perp} = \mathcal{F}$$

where

$$\mathcal{D}^\perp \quad = \quad \big\{x \mid \forall y \in \mathcal{D}, \#(x \cap y) < \omega\big\} \ .$$

Constructions similar to the one above can be defined; and they are characterized by:

- the dual $\mathcal{F}^\perp$;
- $\mathcal{F}_1 \oplus \mathcal{F}_2 = \{x_1 \uplus x_2 \mid x_1 \in \mathcal{F}_1,\ x_2 \in \mathcal{F}_2\}$;
- $\mathcal{F}_1 \otimes \mathcal{F}_2 = \{r \mid \pi_1(r) \in \mathcal{F}_1,\ \pi_2(r) \in \mathcal{F}_2\}$.

Here again, if one looks only at the web $|\mathcal{F}|$ of finiteness spaces, the corresponding operations are just the identity, the usual cartesian product and the disjoint union.

Remarks:

- the constructions on finiteness spaces are actually defined in a way that makes it clear that they yield finiteness spaces. They are latter *proved* to be equivalent what is given above. (See [2].)
- There is another presentation of coherent spaces that closely matches the definition of finiteness spaces: a coherent space is given by a collection $\mathcal{C}$ of subsets of $|\mathcal{C}|$ which satisfy $\mathcal{C}^{\bullet\bullet} = \mathcal{C}$, where $\mathcal{D}^\bullet = \{x \mid \forall y \in \mathcal{D}, \#(x \cap y) \leq 1\}$.

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- Any operator of the form $\mathcal{X} \mapsto \mathcal{X}^\bullet = \{y \mid \forall x \in \mathcal{X},\ R(x,y)\}$ is contravariant and yields a *closure operator* when applied twice.

1. From "coherence" to finiteness. The idea is rather simple: we would like to close the collection of cliques of a coherence space under finite unions. Unfortunately (but unsurprisingly), the notion of "finite unions of cliques" is not very well behaved. We instead consider the following notion:

Definition. *If C is a coherent space, we call a subset of $|C|$* finitely incoherent *if it doesn't contain infinite anticliques. We write $\mathcal{F}(C)$ for the collection of all finitely incoherent subsets of C.*

The following follows directly from the definition:

Lemma.

- *any finite subset of $|C|$ is finitely incoherent;*
- *any clique is finitely incoherent;*
- *a subset of a finitely incoherent subset is finitely incoherent;*
- *finitely incoherent subsets are closed under finite unions.*

Note however that a finitely incoherent set needs not be a finite union of cliques: take for example the graph composed of the disjoint union of all the complete graphs K_n for $n \geq 1$. This graph doesn't contain an infinite clique, but it is not a finite union of anticliques; so, its dual is finitely incoherent but is not a finite union of cliques.

The next lemma is more interesting as it implies that the collection of finitely incoherent subsets forms a finiteness space in the sense of [2]:

Lemma. *If C is a coherence space, we have:*

$$\mathcal{C}(C)^{\perp\!\!\!\perp} = \mathcal{F}(C^\perp) \ .$$

Proof:

($\subseteq$) let x be in $\mathcal{C}(C)^{\perp\!\!\!\perp}$, and suppose, by contradiction, that x is not in $\mathcal{F}(C^\perp)$, *i.e.* x contains an infinite anticlique y of $C^\perp$. This set y is a clique in C, *i.e.* $y \in \mathcal{C}(C)$. Since $x \cap y = y$ is infinite, this contradicts the hypothesis that $x \in \mathcal{C}(C)^{\perp\!\!\!\perp}$.

($\supseteq$) let x be finitely incoherent in $C^\perp$, *i.e.* x doesn't contain an infinite clique of C; let y be in $\mathcal{C}(C)$. Since $x \cap y \in \mathcal{C}(C)$ and $x \cap y$ is contained in x, it cannot be infinite. This shows that $x \in \mathcal{C}(C)^{\perp\!\!\!\perp}$.

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We thus get the expected corollary:

Corollary. *If C is a coherent space, then $\mathcal{F}(C)$ is a finiteness space.*

What was slightly unexpected was the following:

Lemma. *If C is a coherence space, then:*

$$\mathcal{F}(C^\perp) = \mathcal{F}(C)^{\perp\!\!\!\perp} \ .$$

Proof: because of the previous lemma, and because $\_^{\perp\!\!\!\perp}$ is contravariant, we only need to show that $\mathcal{C}(C)^{\perp\!\!\!\perp} \subseteq \mathcal{F}(C)^{\perp\!\!\!\perp}$. Suppose that $x \in \mathcal{C}(C)^{\perp\!\!\!\perp}$, and let $y \in \mathcal{F}(C)$; we need to show that $x \cap y$ is finite.

- Since $x \cap y \subseteq y \in \mathcal{F}(C)$, $x \cap y$ cannot contain an infinite anticlique;
- since $x \cap y \subseteq x \in \mathcal{C}(C)^{\perp\!\!\!\perp}$, $x \cap y$ cannot contain an infinite clique.

Those two points imply, by the infinite Ramsey theorem, that $x \cap y$ is finite.

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The other linear connectives are similarly behaved with respect to the notion of finitely incoherent sets. We have:

Lemma. *If C_1 and C_2 are coherent spaces, then we have both*

$$\mathcal{F}(C_1 \oplus C_2) \quad = \quad \mathcal{F}(C_1) \oplus \mathcal{F}(C_2) \ ,$$

and

$$\mathcal{F}(C_1 \otimes C_2) \quad = \quad \mathcal{F}(C_1) \otimes \mathcal{F}(C_2)$$

where the connectives on the left are the coherent spaces ones, and the connectives on the right are the finiteness ones.

Proof: the $\oplus$ part is direct; for the $\otimes$ part, recall that $r \in \mathcal{F}(C_1) \otimes \mathcal{F}(C_2)$ is equivalent to $\pi_1(r) \in \mathcal{F}(C_1)$ and $\pi_2(r) \in \mathcal{F}(C_2)$.

($\subseteq$) suppose r doesn't contain an infinite anticlique; neither $\pi_1(r)$ nor $\pi_2(r)$ can contain an infinite anticlique, as it would imply the existence of an infinite anticlique in r.

($\supseteq$) suppose that $r \in \mathcal{F}(C_1) \otimes \mathcal{F}(C_2)$ contains an infinite anticlique r' of $C_1 \otimes C_2$. At least one of $\pi_1(r')$ or $\pi_2(r')$ must be infinite, otherwise, r' itself would be finite. Suppose $\pi_1(r')$ is infinite; because $\pi_1(r') \subseteq \pi_1(r)$, it cannot contain an infinite anticlique. By the infinite Ramsey theorem, it thus contains an infinite clique x. For each $a \in x$, chose one element b inside the "fiber" $r'(a) = \{b \mid (a,b) \in r'\}$. Two such b's can't be coherent as it would contradict the fact that r' is an anticlique: we have constructed an infinite anticlique in $\pi_2(r')$. Contradiction!

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Note that because the same is true of finiteness spaces, the previous corollary implies in particular that

$$\mathcal{F}\left((C_1 \oplus C_2)^\perp\right) \quad = \quad \mathcal{F}\left(C_1^\perp \oplus C_2^\perp\right) \ .$$

The direct proof of this equality is also quite easy.

Both coherent spaces and finiteness spaces form categories, where:

- a morphism from C to D in **Coh** is a clique in $(C \otimes D^\perp)^\perp$,
- a morphism from $\mathcal{F}$ to $\mathcal{G}$ in **Fin** is a finitary set in $(\mathcal{F} \otimes \mathcal{G}^{\perp\!\!\!\perp})^{\perp\!\!\!\perp}$.

In both cases, morphisms are special relations between the webs and composition is the usual composition of relations. From all the above, we can conclude that:

Proposition. *$\mathcal{F}(\_)$ can be lifted to a functor from* **Coh** *to* **Fin***:*

- *it sends C to $\mathcal{F}(C)$*
- *and $r \in \mathbf{Coh}[C,D]$ to $r \in \mathbf{Fin}[\mathcal{F}(C), \mathcal{F}(D)]$.*

Moreover, this functor commutes with the logical connectives.

Note that this functor is faithful (but not full); and that it is not injective on objects: adding and removing any finite number of edges to a coherence space doesn't change its image via $\mathcal{F}(\_)$.

In a sense, coherence spaces allow to define a collection of "simple" finiteness spaces. An informal argument regarding this "simplicity" can be found in the following remark: the logical complexity of the formula expressing $x \in \mathcal{A}^{\perp\!\!\!\perp\perp\!\!\!\perp}$ for a lower-closed collection of subsets $\mathcal{A}$. In the general case (see [2]), we have

$$x \in \mathcal{A}^{\perp\!\!\!\perp\perp\!\!\!\perp} \quad \Leftrightarrow \quad \forall y \subseteq x, \#(y) = \infty \ \exists z \subseteq y, \#(z) = \infty \ z \in \mathcal{A} \ .$$

Because $\#(y) = \infty$ is a $\mathsf{\Sigma}^1_1$-formula (only existential second order quantifiers), this is a $\mathsf{\Pi}^1_2$-formula (second order quantifiers are $\forall\exists$). For the particular case when $\mathcal{A}$ is the set of cliques of a coherent spaces C, we obtain

$$x \in \mathcal{A}^{\perp\!\!\perp\perp\!\!\perp} \quad \Leftrightarrow \quad \forall y \subseteq x, \#(y) = \infty \; \exists a, b \in y \; (a, b) \in C$$

which is only a $\mathsf{\Pi}^1_1$-formula.

$1\frac{1}{2}$. Cardinality of finiteness spaces. So, coherence spaces can be used to define a (non-full) subcategory of finiteness spaces, closed under the logical operations ($\_^{\perp}$, $\_\otimes\_$ and $\_\oplus\_$). It is natural to ask whether all finiteness spaces can be obtained in this way. The previous remark about the logical complexity of coherence versus finiteness points toward a negative answer. Here is a more formal proof, which also answers a question raised by T. Ehrhard:

Proposition. *If A is infinite countable, the cardinality of finiteness spaces on A is exactly that of $\mathcal{P}(\mathcal{P}(A))$. The cardinality doesn't change if we consider finiteness on A "up to permutations".*

Since the cardinality of coherence spaces on A is the same as that of $\mathcal{P}(A \times A) \simeq \mathcal{P}(A)$, we have:

Corollary. *If A is infinite countable, there are strictly more finiteness spaces on A than coherence spaces on A.*

Proof of the proposition: let A be infinite countable; "up to isomorphism", we can assume that $A = \mathbf{B}^{<\omega}$, the set of finite sequences of bits. If x is an infinite sequence of bits (a non dyadic real), write $x^{\downarrow}$ for the set of finite approximations of x; and if X is a set of such "real numbers", write $X^{\downarrow}$ for the set $\{x^{\downarrow} \mid x \in X\}$. We have $X^{\downarrow} \subset \mathcal{P}(A)$ for any such X.

Suppose now that $X \neq X'$ with, for example, $x \in X$ but $x \notin X'$. Since $x^{\downarrow}$ is infinite and $x^{\downarrow} \in X^{\downarrow}$, we have $x \notin X^{\downarrow\perp\!\!\perp}$. However since two different reals must differ on some finite approximation, we have that $x^{\downarrow} \in X'^{\downarrow\perp\!\!\perp}$. Thus, the finiteness spaces $(A, X^{\downarrow\perp\!\!\perp})$ and $(A, X'^{\downarrow\perp\!\!\perp})$ differ.

Thus, finiteness spaces on A have the same cardinality as $\mathcal{P}(\mathbf{R}) \simeq \mathcal{P}(\mathcal{P}(A))$.

The cardinality of finiteness spaces on A *modulo permutation on* A is the same because equivalence classes are of cardinality at most $\#(\mathcal{P}(A))$ (there are exactly $\#\big(\mathcal{P}(A)\big)$ permutations on A), and since $\kappa \times \#\big(\mathcal{P}(A)\big) = \max\big(\kappa, \#(\mathcal{P}(a))\big)$, there must be at least $\#\big(\mathcal{P}(\mathcal{P}(A))\big)$ such equivalence classes to cover the whole collection of finiteness spaces.

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It is slightly interesting to note that the same reasoning doesn't apply to higher cardinalities since $\#(A^{<\omega}) = \#(A)$ if A is uncountable.

ω. Conclusion. The situation with respect to full linear logic isn't totally clear. We have a base category $\mathbf{F}$ with:

- coherence spaces as objects
- and finitely incoherent linear maps as morphisms: $\mathbf{F}[C, D] = \mathcal{F}\big((C \otimes D^{\perp})^{\perp}\big)$.

This category is a linear, full subcategory of the category **Fin** of finiteness spaces.

Lifting the usual notion set-based of exponentials for coherence spaces to this category is impossible: because of uniformity (considering only the cliques of C as vertices of $\mathbf{!}C$), this construction isn't even functorial. Take for example K_n and $K_n^{\perp}$: they are isomorphic in $\mathbf{F}$ but $\mathbf{!}K_n$ and $\mathbf{!}(K_n^{\perp})$ cannot be because their webs have different cardinalities, namely 2^n and $n{+}1$. A similar phenomenon is expected if we use the multiset-based notion of exponentials. Using non-uniform coherence spaces ([1]) isn't a solution because Ramsey theorem cannot be used anymore. Finding an appropriate notion of exponential to extend this category to a model of the algebraic λ-calculus ([5]), or better yet, of the differential λ-calculus by is thus left open at this point.

References

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